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A Note on the Approximation of Empirical Critical Values - An Application to Tests of Exponentiality

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Abstract - The behaviour of empirical critical values in asymptotic tests for exponentiality with respect to their protection against errors of type – I and type – II when approximated by exact theoretical values is discussed. Three tests for exponentiality whose statistics are normally distributed are arbitrarily selected for the study. Each of the tests is described. The empirical critical values of the tests are obtained through Monte Carlo computations under different levels of significance and different sample sizes and these values are compared with their corresponding theoretical critical values. The results are discussed and interpreted.

Keywords **- Empirical Critical Value, Goodness – of – Fit Test, Monte Carlo Simulation, Test for Exponentiality, Type – I – Error, Type – II – Error.**

1. Introduction

It is a common practice among some users of statistics to assume that a set of data $x_1, x_2, ..., x_n$, for a statistical inference comes from a certain distribution without a formal statistical test to ascertain the correctness or otherwise of the claim. This is very risky because the robustness of most techniques employed in statistical inference cannot be guaranteed and as a result such tests may be very sensitive to underlying distribution of the data set. In the event of this, the outcome of such a statistical inference is invalidated with wrong distributional assumption, the consequence of which varies with study. For instance, in disease control and reliability studies where investigations are carried out with utmost carefulness, a misleading result owing to wrong distributional assumption goes with a dare consequence.

Suppose a sample of n independent and identically distributed (i.i.d) observations, $x_1, x_2, ..., x_n$ are available from an unknown distribution function $F(\underline{x})$, the goodness – of – fit test of the set of data to a certain distribution with distribution function $F_0(x)$ and density function $f_0(x)$ involves testing the hypothesis $H_0 : F(\underline{x}) = F_0(\underline{x})$ against the alternative $H_1 : F(\underline{x}) \neq F_0(\underline{x})$, see [1]. Several procedures have been developed in the literature for this purpose according to certain characterizations of the suspect distribution, $F_0(x)$. A good number of these procedures for different distributions are exact, having test statistics with exact probabilities according to some well known probability distributions. Also, some of the procedures are asymptotic, having test statistics with approximate probabilities according to some well known probability distributions. Yet, other procedures exist with test statistics which cannot be approximated by any known probability distribution. While the applicability of the test procedures of this class of tests depends solely on empirical critical values and that of the exact class of tests depends on exact critical values, the applicability of the class of asymptotic tests of goodness – of – fit depends on either of the two critical values which are expected to be approximately equal.

One distribution that underlies a good number of statistical studies in health sciences, especially those that are related to renewal process, birth and death process, markov process, queuing theory and every other process characterized with appreciation and/or progressive decay (ageing) and reliability studies as well as survival analysis and generally in modeling is the exponential distribution.

Goodness-of-fit test for exponentiality of a data set has been discussed extensively in the literature by a good number of authors including the Pearson's chi-square test of goodness-of-fit, [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14] and [15]. These tests are based on certain characterizations of the exponential distribution such as empirical distribution function (EDF), empirical characteristic function (ECF), empirical Laplace transform (ELT), normalized spacing, residual entropy and mean residual life functions.

Let $x_1, x_2, ..., x_n$, be a set of observations from an unknown distribution $F(x)$ with probability function $f(x)$. The problem of testing for exponentiality involves testing the goodness-of-fit hypothesis, $H_0 : F(x)$ is an exponential distribution, against the alternative, $H_1 : F(x)$ is not an exponential distribution.

The ability of the various tests for exponentiality to reject the null hypothesis when the true distribution of the data set is not exponential, known as the power of the test, varies from one test to another and depends on the nature of the true distribution of the data set, see [16]. Compared to the amount of efforts that have been put into development of several tests for exponentiality, much work has not been devoted to assessing the quality of these tests with respect to their power. [16] compares the powers of fifteen different tests for exponentiality against three classes of distributions, namely: alternative distributions with increasing hazard rate (IHR), alternative distributions with decreasing hazard rate (DHR) and alternative distributions with non-monotone hazard rate $(NMHR)$. Of all the tests considered, the statistics according to [5], [4], Kolmogorov-Smirnov, [17] among others were adjudged good against alternatives with monotone hazard rate (increasing or decreasing hazard rate). For the non-monotone hazard rate alternatives, the tests according to [5], [2] and [3] were found to be competitive. However, in most practical cases, the nature of the distribution is usually not known before the goodness – of – fit test and as a result, [16] generally recommends [5].

In a similar way, [18] in a partial review of tests for exponentiality compares the old and newer tests for exponential distribution, discarding less powerful ones according to [16]. Also, [19], in a broader work which includes over eighteen different procedures, agreed in part with [16] as well as with [18]. While [16] and [19] comparisons are based on the class of the alternative distributions in terms of the monotonicity of hazard rate, [18] tends to be compared along classes of competing tests based on the characterizations of the exponential distribution.

Some of the statistics developed in the various tests for exponentiality have either exact or asymptotic distribution; see for instance [16]. They are employed in the tests by using the theoretical critical values of their exact or approximate distributions. Also, a good number the statistics developed for a similar purpose are applied via empirical critical values owing to the fact that their exact or asymptotic distributions cannot be determined, at least in an applicable form, see for instance [18]. In this class of statistics for exponentiality test, the empirical critical value for each sample size is determined, at a given level of significance, through Monte Carlo simulation. This is done by obtaining a very large number of equal – sized samples from the null distribution and computing from each of the samples the realization of the statistic in question so as to obtain the appropriate fractile that agrees with the stated level of significance as the empirical critical value.

Obviously, the volume of computation involved in the determination of empirical critical values is enormous. However, it is possible to get them obtained for each test at different sample sizes and levels of significance so that they can be made available for easy applicability of the tests. The problem of standard error of estimates in the computation of the empirical critical values expectedly decreases with increase in the number of repetitions in simulation.

In other words, for a sufficiently large number of simulations, it would ordinarily be expected that the observed difference in the power of the test attributed to the use of empirical critical value instead of the theoretical one would be insignificant. This observed difference between the empirical and corresponding theoretical critical values is exactly the problem which the paper seeks to address.

This study is considered very imperative especially now that the literature in goodness – of – fit tests is generally experiencing remarkable growth in this direction of test statistics with empirical critical values. Some well regarded tests for exponentiality whose statistics are known to have asymptotic distributions are arbitrarily selected for the study. Section 2 gives a description of the statistics to be used as well as their asymptotic distributions. In section 3, the simulation study will be carried out while discussion of results and conclusion are given in section 4.

2. Description of Tests

In this section, three tests shall be used for the above problem. They are the tests developed in [5]; [20], and [6], which shall be denoted by CO_n, G_n and EP_n respectively. They are all asymptotic and are selected arbitrarily for the study among all the asymptotic tests for exponentiality. In what follows therefore, their descriptions shall be given.

2.1 The CO_n Test

The test in [5] is a two-sided test of exponentiality whose statistic is given as:

$$CO_n = n + \sum_{j=1}^{n}\left(1 - Y_j\right) In Y_j;\ j = 1,2,...,n;\ Y_j = X_j / \bar{X}_n,$$ and

$$\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j \quad (2.1)$$

The statistic rejects the null hypothesis of exponentiality for both small and large values of CO_n. It is also obtained that the asymptotic null distribution of the statistic is such that

$$CO_n^* = \left(6/n\right)^{1/2}\left(CO_n / \pi\right) \sim N(0,1) \quad (2.2)$$

As a result of the asymptotic null distribution, it is possible to use standard normal critical values when conducting a test of exponentiality via this statistic.

2.2 The G_n Test

In [20], a scale – free goodness – of – fit test for exponentiality is studied which is based on the normalized spacing and a rejection of the null hypothesis is proposed by the test for large values of the statistic:

$$G_n = \frac{1}{2n(n-1)}\sum_{j,k=1}^{n}\left|Y_j - Y_k\right|; \quad j,k = 1, 2, ..., n; Y_{j,k} = X_{j,k} / \bar{X}_n \quad (2.3)$$

It also obtained that the asymptotic null distribution of the statistic:

$$G_n^* = \left\{12(n-1)\right\}^{1/2}\left(G_n - 1/2\right) \quad (2.4)$$

is standard normal even for samples as small as $n = 10$ and claimed that the statistic has very high power against Weibull, Uniform and Gamma alternatives.

2.3 The EP_n Test

An exponential random variable with parameter θ has a characteristic function $\varphi(t) = 1/(1 - i\theta t)$ with a parametric estimator $\hat{\varphi}(t) = 1/(1 - i\bar{X}_n t)$, where $\bar{X}_n = (1/n)\sum_{j=1}^{n} X_j$ is the sample mean. Comparing this with the empirical characteristic function, ECF, of an exponential random sample, [6] proposed a statistic:

$$EP_n = \left(48n\right)^{1/2}\int_{-\infty}^{\infty}\left\{\varphi_n(t) - \frac{1}{(1 - i\bar{X}_n t)}\right\}\frac{\bar{X}_n}{2\pi(1 + i\bar{X}_n t)}dt \quad (2.5)$$

Straight forward integration and simplification of eq. (2.5) gives it in a more computational form as:

$$EP_n = \left(48n\right)^{1/2}\left\{\frac{1}{n}\sum_{j=1}^{n}\exp\left(-Y_j\right) - \frac{1}{2}\right\}; \quad j = 1, 2, ..., n \quad (2.6)$$

and $Y_j = X_j / \bar{X}_n$ as usual. The asymptotic distribution of EP_n statistic is standard normal and the null hypothesis of exponentiality is rejected for large values of EP_n. The test is consistent against any fixed alternative distribution with monotone hazard rate, provided the distribution is absolutely continuous.

3. Simulation Study

In this section, comparisons of the empirical and theoretical critical values in asymptotic tests for exponentiality with respect to their type one and type two error rates shall be made by using Monte Carlo computations. The theoretical critical values, adapted from [21], for 0.005, 0.01, 0.025 and 0.05 levels of significance are presented in table 1.The empirical critical values of the three tests for 20, 30, 50 and 100 sample sizes are obtained, each at 0.005, 0.01, 0.025 and 0.05 levels of significance by 5,000 simulations. These values are presented in Table 2. Under each test for exponentiality considered here, 5,000 samples of sizes *n* = 20, 30, 50, and *n* = 100 are generated and for each sample, the statistics are evaluated. The empirical critical value is obtained from each statistic at a given level of significance, α and sample size, *n* as the 100(1 - α)th percentile of the 5,000 Monte Carlo computations. Each average value in table 2 is obtained as the average empirical critical value of all the 4 sample sizes (n = 20, 30, 50, and *n* = 100) considered for a level of significance.

4. Discussion of Results and Conclusion

The exact critical values for a standard normal distribution, which the tests, CO_n^*, G_n^*, and EP_n, set to approximate are presented in table 1 for different levels of significance while the corresponding empirical critical values for sample sizes n =20, 30, 50 and *n* = 100 are presented in table 2. It is well known that an appreciation of the critical value of a test beyond the expected (exact critical value) gives room for a wrong null hypothesis to be accepted, (type – II – error) while a depreciation of the critical value beyond the expected tends to allow rejection of a null hypothesis which is right, (type – I – error).

Table 1. The theoretical (exact) critical values of the standard normal distribution at different levels of significance

α	0.050	0.025	0.010	0.005
Exact critical values	1.645	1.960	2.325	2.578

Based on these, the empirical critical values of the tests for exponentiality considered here (CO_n^*, G_n^*, and EP_n) are expected to be very close to the corresponding exact critical values of the standard normal distribution to give the needed protection to the tests against the errors of type – I and type – II.

From table 2, we can see that the tests vary in their approximations to the standard normal distribution. The CO_n^* test maintains a fair decrease of the type – II – error from sample size n = 20 to the one of n = 100 in almost all the levels of significance without a check on the increase of type – I – error. The tests G_n^* and EP_n, on the other hand, maintain a fair reduction of type – I – error with an appreciable check on the type – II – error. The average critical values in all the three tests show more protection against the two errors for all the levels of significance considered. It can be seen that the deviations of the empirical critical values from the theoretical ones are minimum with the average values.

Table 2. The empirical critical values of tests for exponentiality (α = 0.005, 0.01, 0.025 and 0.05; n = 20, 30, 50 and 100)

n	α	CO_n	CO_n^*	G_n	G_n^*	EP_n
20	0.050	10.0512	1.7495	0.6047	1.6202	1.3842
	0.025	10.6972	1.8940	0.6254	1.9339	1.7831
	0.010	11.8277	2.1606	0.6283	2.1469	2.1218
	0.005	12.1127	2.1610	0.6352	2.1470	2.1220
30	0.050	12.2572	1.5672	0.5638	1.4432	1.4044
	0.025	12.7176	1.8349	0.5690	1.9020	1.7116
	0.010	14.8897	2.2983	0.6149	2.2863	2.3668
	0.005	14.8900	2.2990	0.6150	2.3000	2.5670
50	0.050	13.2332	1.4586	0.5759	1.8410	1.4343
	0.025	15.9652	1.7597	0.5873	2.0291	2.0036
	0.010	21.9220	2.4163	0.5978	2.3731	2.3840
	0.005	21.9230	2.8269	0.6052	2.5519	2.5700
100	0.050	17.3053	1.3487	0.5403	1.4283	1.2945
	0.025	20.0142	1.6154	0.5460	1.7142	1.6286
	0.010	22.0604	1.7082	0.5519	2.0270	1.7928
	0.005	29.1656	2.8269	0.5553	2.0280	1.9781
Average	0.050	13.2117	1.5310	0.5712	1.5782	1.3794
	0.025	14.8486	1.7760	0.5819	1.8948	1.7817
	0.010	17.6750	2.1459	0.5982	2.2083	2.1664
	0.005	19.5228	2.4260	0.6027	2.2565	2.3093

Based on the observations made so far, it is very likely that tests for exponentiality, with asymptotic test statistics, based on exact critical values are vulnerable to errors of type – I and type – II. It is therefore important to recommend here that such test statistics should be applied in this respect with a caution. In a situation where the empirical critical value of such a test statistic is known, it is safer to use it instead of the theoretical (exact) critical values of the distribution.

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